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How to tune the system parameters to realize stochastic resonance

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Abstract
The paper presents a numerical method of realizing stochastic resonance (SR) by tuning system parameters. Firstly, a simple and effective method of evaluating the system response speed λ_1 is introduced. Then the parameter-induced SR problem can be reduced to the parameter optimization problem under the condition $\lambda_1 = \text{const}$. To solve this optimization problem, we put forward the following techniques: (i) compensate the deviation from the prescribed system response speed λ_1 at each step, (ii) take the curve length increment on the parameter plane as the step size in searching the maximal signal-to-noise ratio (SNR) gain. Finally, a numerical simulation confirms the effectiveness of this method.

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1. Introduction

Stochastic resonance (SR) was originally reported in 1981 as a mechanism that might explain why the Earth's ice ages occur more or less periodically every 100 000 years [1]. Since its introduction, this phenomenon has been shown in optical systems, biological systems, chemical systems, etc. We may refer to the very exhaustive review [2]. Furthermore, the study of SR in signal analysis has received considerable attention in recent years [3–18].

Essentially, SR is based on the cooperative effects between the stochastic-subjected dynamical system and the external periodic forcing [1]. SR can be realized by tuning the stochastic-subjected dynamical system or varying the forcing frequency [19, 20]. To tune the stochastic-subjected dynamical system, besides adding noise, we can tune the intrinsic characteristic of the dynamical system.

According to the notion of classical SR, its occurrence needs three ingredients: a double well potential, periodic forcing and a noise. So the dynamical system can be modelled by the differential equation

$$\dot{x}(t) = ax(t) - \mu x^3(t) + h(t) + \xi(t) \qquad a, \mu > 0 \tag{1}$$

where a and μ are system parameters, $h(t)$ is the input periodic signal, and $\xi(t)$ approximates the exponentially correlated coloured noise with Gaussian distribution and zero mean. Such noise can be generated by a Gauss–Markov process via the following equation,

$$\tau\dot{\xi}(t) = -\xi(t) + \Gamma(t) \tag{2}$$

where τ is the noise correlation time, and $\Gamma(t)$ is a zero mean Gaussian white noise with the noise intensity D. This model was introduced in [7], based on a popular work appeared in [21]. Here, tuning the intrinsic characteristic of the dynamical system means tuning the system parameters a and μ. In previous papers [14–16], we demonstrated that when the noise intensity and the forcing frequency were fixed, SR could be realized by tuning the system parameters. In fact, SR realized by tuning the system parameters first appeared in the research of SR in Chua's circuit [17]. It was also shown in [18] that there are optimal values for the threshold in the detection of noisy signals with neuron-like threshold crossing detectors. In a review paper [3], the fact that SR can be realized by adjusting system parameters was emphasized in the signal processing field.

In section 2, we will introduce a method of evaluating the system response speed λ_1 because it is very important for tuning system parameters. The inverse of the system response speed $(1/\lambda_1)$ is the characteristic time of the system, which dominates the time of the system tending to the steady state. The input signal we considered in this paper is periodic. It was demonstrated that the ac signal can be viewed as the connections of dc signals with different levels when the system response speed is high enough [23]. Therefore, for the convenience of calculation, $h(t)$ in equation (1) is reduced to a constant h in sections 2 and 3. Moreover, we hope that the system response speed is high enough in order to trace the varying of the input signal. Section 3 presents the method of tuning system parameters in detail. This method can be briefly described as follows: (i) assume that the noise correlation time τ, the noise intensity D and the input signal h are fixed; (ii) according to the highest frequency of the input signal, select a large enough system response speed; (iii) search the maximal signal-to-noise ratio (SNR) gain by tuning system parameters a and μ with the fixed system response speed. In fact, the parameter-induced SR problem can be reduced to the parameter optimization problem under the condition $\lambda_1 = \text{const}$. To solve this optimization problem, we put forward two techniques: (i) compensate the deviation from the prescribed system response speed at each step; (ii) take the curve length increment on the parameter plane as the step size in searching the maximal SNR gain. In section 4, by a numerical simulation, the notion of SR is applied in order to denoise a periodic signal, and the simulation results confirm the effectiveness of the optimization. The conclusions are given in section 5.

2. The solution of the system response speed

Rescale the variables according to

$$\bar{\tau} = a\tau \qquad \bar{t} = at \qquad y = \sqrt{\frac{a}{D}}x \qquad \bar{\mu} = \frac{\mu D}{a^2} \qquad \bar{h} = \frac{h}{\sqrt{aD}}. \tag{3}$$

Equations (1) and (2) can be rewritten as

$$\dot{y}(\bar{t}) = y(\bar{t}) - \bar{\mu}y^3(\bar{t}) + \bar{h} + \xi(\bar{t}) \tag{4}$$

$$\bar{\tau}\dot{\xi}(\bar{t}) = -\xi(\bar{t}) + \Gamma(\bar{t}). \tag{5}$$

Obviously, the number of the parameters is decreased and the noise intensity becomes unity.

The system response speed $\bar{\lambda}_1$ can be evaluated from the following formula [23]

$$\bar{\lambda} = \underset{F\neq 0}{\mathrm{st}} \frac{\int_{-\infty}^{+\infty} F'^2(y)\exp[g(y)]\,\mathrm{d}y}{\int_{-\infty}^{+\infty}[1-\bar{\tau}\bar{c}'(y)]F^2(y)\exp[g(y)]\,\mathrm{d}y} \tag{6}$$

where st means the stationary value in variational problems, and

$$\bar{c}(y) = y - \bar{\mu}y^3 + \bar{h} \tag{7}$$

$$\bar{c}'(y) = 1 - 3\bar{\mu}y^2 \tag{8}$$

$$g(y) = \int_0^y \bar{c}(y)\,\mathrm{d}y - \frac{1}{2}\bar{\tau}\bar{c}^2(y). \tag{9}$$

Equation (6) is demonstrated in appendix A in detail.

In the following we solve the variational problem (6) numerically by the Hermite interpolating method. In the range of $y \in [y_k, y_{k+1}]$, perform the coordinate transformation

$$y = y_k + \tfrac{1}{2}(y_{k+1} - y_k)(\zeta + 1) \qquad -1 \leqslant \zeta \leqslant 1. \tag{10}$$

Then the Hermite interpolating functions of $F(\zeta)$ and $F'(\zeta)$ with $-1 \leqslant \zeta \leqslant 1$ are

$$F(\zeta) = \sum_{i=1}^{4} f_i(\zeta)F_i = [f_1 \quad f_2 \quad f_3 \quad f_4]\{F^e\} \tag{11}$$

$$F'(\zeta) = \sum_{i=1}^{4} f_i'(\zeta)F_i = [f_1' \quad f_2' \quad f_3' \quad f_4']\{F^e\} \tag{12}$$

where

$$\begin{aligned} &f_1(\zeta) = 0.25(2+\zeta)(1-\zeta)^2 && f_1'(\zeta) = -0.75(1-\zeta)(1+\zeta) \\ &f_2(\zeta) = 0.25(1+\zeta)(1-\zeta)^2 && f_2'(\zeta) = -0.25(1-\zeta)(1+3\zeta) \\ &f_3(\zeta) = 0.25(2-\zeta)(1+\zeta)^2 && f_3'(\zeta) = 0.75(1-\zeta)(1+\zeta) \\ &f_4(\zeta) = -0.25(1-\zeta)(1+\zeta)^2 && f_4'(\zeta) = -0.25(1+\zeta)(1-3\zeta) \\ &\{F^e\} = [F_1 \quad F_2 \quad F_3 \quad F_4]^T \end{aligned}$$

in which

$$F_1 = F(y_k) \qquad F_3 = F(y_{k+1}) \qquad F_2 = F'(y_k) \qquad F_4 = F'(y_{k+1}).$$

From equation (10), $\mathrm{d}F/\mathrm{d}y$ is given by

$$\frac{\mathrm{d}F}{\mathrm{d}y} = F'(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}y} = \frac{2}{y_{k+1} - y_k}F'(\zeta) = \frac{2}{\Delta y_k}F'(\zeta). \tag{13}$$

With the help of equations (10)–(13), we have

$$\begin{aligned} \int_{y_k}^{y_{k+1}} F^2(y)[1-\bar{\tau}\bar{c}'(y)]\exp[g(y)]\,\mathrm{d}y &= \int_{-1}^{1} F^2(\zeta)[1-\bar{\tau}\bar{c}'(\zeta)]\frac{\Delta y_k}{2}\exp[g(\zeta)]\,\mathrm{d}\zeta \\ &= \{F^e\}^T[M^e]\{F^e\} \end{aligned} \tag{14}$$

$$\int_{y_k}^{y_{k+1}} F'^2(y)\exp[g(y)]\,\mathrm{d}y = \int_{-1}^{1} \frac{2}{\Delta y_k} F'^2(\zeta)\exp[g(\zeta)]\,\mathrm{d}\zeta$$
$$= \{F^e\}^T[K^e]\{F^e\} \tag{15}$$

where the element matrices $[M^e]$ and $[K^e]$ are

$$[M^e] = \left[M^e_{ij}\right] \qquad [K^e] = \left[K^e_{ij}\right] \qquad i, j = 1, 2, 3, 4 \tag{16}$$

in which

$$M^e_{ij} = \int_{-1}^{1} \frac{\Delta y_k}{2}[1 - \bar{\tau}\bar{c}'(\zeta)] f_i f_j \exp[g(\zeta)]\,\mathrm{d}\zeta \tag{17}$$

$$K^e_{ij} = \int_{-1}^{1} \frac{2}{\Delta y_k} f_i' f_j' \exp[g(\zeta)]\,\mathrm{d}\zeta. \tag{18}$$

By uniting the element matrices as in the usual finite element method, we have

$$\int_{-\infty}^{+\infty} [1 - \bar{\tau}\bar{c}'(y)]F^2(y)\exp[g(y)]\,\mathrm{d}y = \{F\}^T[\bar{M}]\{F\} \tag{19}$$

$$\int_{-\infty}^{+\infty} F'^2(y)\exp[g(y)]\,\mathrm{d}y = \{F\}^T[\bar{K}]\{F\}. \tag{20}$$

Because F_2, F_4 etc are derivatives with respect to ζ, transform them into derivatives with respect to y, i.e.,

$$[M] = [\mathrm{Tr}]^T[\bar{M}][\mathrm{Tr}] \qquad [K] = [\mathrm{Tr}]^T[\bar{K}][\mathrm{Tr}] \tag{21}$$

where

$$[\mathrm{Tr}] = \mathrm{diag}\left[1 \quad \tfrac{\Delta y_0}{2} \quad \cdots \quad 1 \quad \tfrac{\Delta y_{n-1}}{2}\right].$$

In equation (6) the integral upper limit and lower limit can be estimated by $\int_{-\infty}^{y_0} \exp[0.5g(y)]\,\mathrm{d}y$ and $\int_{y_n}^{+\infty} \exp[0.5g(y)]\,\mathrm{d}y$, respectively. For example, if the precision defined is 0.001, y_0 and y_n are obtained by the following equations,:

$$\int_{-\infty}^{y_0} \exp[0.5g(y)]\,\mathrm{d}y = 0.001 \qquad \int_{y_n}^{+\infty} \exp[0.5g(y)]\,\mathrm{d}y = 0.001.$$

From equations (19)–(21), equation (6) can be rewritten as

$$([K] - \bar{\lambda}[M])\{F\} = 0. \tag{22}$$

Obviously, the matrices $[M]$ and $[K]$ are both symmetric. From equation (6), $[M]$ is positive and $[K]$ is semi-positive. The system response speed $\bar{\lambda}_1$ is the minimum positive eigenvalue [14]. In our previous method [14], $F(y)$ in equation (6) is a polynomial function, not a piecewise interpolating function. That will lead to the large difference among the elements of the matrix $[M]$ or $[K]$ and make the matrices ill-conditioned. Hence, it will bring on difficulties in evaluating the system response speed. However, the Hermite interpolating method can avoid this case and obtain a more accurate solution.

3. The method of tuning system parameters

To obtain the maximal SNR gain (i.e., realize SR), it is necessary to optimize the system parameters. In this section, we will discuss this in detail.

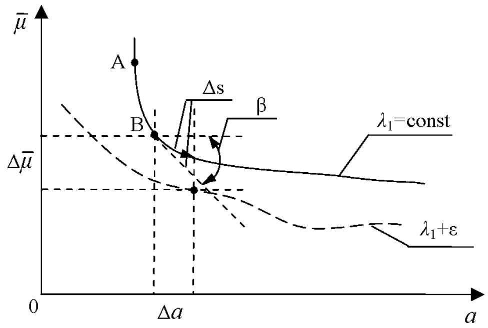

Figure 1. A sketch of a versus $\bar{\mu}$. a and $\bar{\mu}$ satisfy the prescribed system response speed.

3.1. The description of the problem

The SNR gain is discussed in appendix B. From equations (3), (B.7) and (B.8), $\mathrm{SNR}_{\mathrm{out}}$ can be expressed as

$$\mathrm{SNR}_{\mathrm{out}} = \mathrm{SNR}_{\mathrm{out}}(\bar{h}, \bar{\mu}, \bar{\tau}) = \mathrm{SNR}_{\mathrm{out}}(\mathrm{SNR}_{\mathrm{in}}, a, \bar{\mu}, \tau). \tag{23}$$

The system response speed $\bar{\lambda}_1$ evaluated from equation (6) is the inverse of the characteristic time of the system (4). According to the scale equation (3), the practical system response speed is

$$\lambda_1 = a\bar{\lambda}_1(\bar{h}, \bar{\mu}, \bar{\tau}) = a\bar{\lambda}_1(\mathrm{SNR}_{\mathrm{in}}, a, \bar{\mu}, \tau). \tag{24}$$

As described in section 1, $\mathrm{SNR}_{\mathrm{in}}$ and τ are prescribed, so equations (23) and (24) are rewritten as

$$\mathrm{SNR}_{\mathrm{out}} = \mathrm{SNR}_{\mathrm{out}}(a, \bar{\mu}) \tag{25}$$

and

$$\lambda_1 = a\bar{\lambda}_1(a, \bar{\mu}). \tag{26}$$

The system response speed λ_1 should be determined by the highest frequency of the input signal. Note that it should be high enough in order to trace the varying input signal. However, too high speeds cannot be chosen, otherwise the stochastic deviation of the output will increase. In fact, by our recent research, there is an optimal system response speed, which is related to the frequencies and the amplitude of the input signal and the noise intensity. It will be discussed in detail in subsequent papers. Here, we take the system response speed 4–8 times the highest frequency of the input signal. Under the condition $\lambda_1 = \mathrm{const}$, tune the parameters a and $\bar{\mu}$ until a maximal SNR gain is obtained, i.e., realize SR by tuning the parameters a and $\bar{\mu}$. According to equation (B.9), the SNR gain is maximal with the maximal output SNR. Therefore, the parameter optimization problem can be expressed as

$$\max_{\substack{a,\bar{\mu} \\ \tau,\mathrm{SNR}_{\mathrm{in}},\lambda_1=\mathrm{const}}} \mathrm{SNR}_{\mathrm{out}}(a, \bar{\mu}) = \mathrm{SNR}_{\mathrm{out}}^{\max}(\tau, \mathrm{SNR}_{\mathrm{in}}, \lambda_1). \tag{27}$$

Figure 1 is a sketch of a versus $\bar{\mu}$ when they satisfy the prescribed system response speed λ_1. The optimal a and $\bar{\mu}$ should be searched along the solid curve.

In the following, we explore the equation satisfied by δa and $\delta\bar{\mu}$ under the condition $\lambda_1 = \mathrm{const}$.

From equation (26), the condition $\lambda_1 = \mathrm{const}$ can be rewritten as

$$\delta\lambda_1 = \bar{\lambda}_1\delta a + a\delta\bar{\lambda}_1 = 0. \tag{28}$$

To obtain the expression for $\delta\bar{\lambda}_1$, first deal with the following variation,

$$\delta\frac{\int_{-\infty}^{+\infty} F'^2(y)\exp[g(y)]\,\mathrm{d}y}{\int_{-\infty}^{+\infty}[1-\bar{\tau}\bar{c}'(y)]F^2(y)\exp[g(y)]\,\mathrm{d}y} = \delta L = \frac{A}{B} - L\cdot\frac{C}{B} \tag{29}$$

where

$$A = \int_{-\infty}^{+\infty} \delta g(y)F'^2(y)\exp[g(y)]\,\mathrm{d}y \tag{30}$$

$$B = \int_{-\infty}^{+\infty} [1-\bar{\tau}\bar{c}'(y)]F^2(y)\exp[g(y)]\,\mathrm{d}y \tag{31}$$

$$C = \int_{-\infty}^{+\infty} \{[1-\bar{\tau}\bar{c}'(y)]\delta g(y) - \tau\bar{c}'(y)\delta a - \bar{\tau}\delta\bar{c}'(y)\}F^2(y)\exp[g(y)]\,\mathrm{d}y \tag{32}$$

$$L = \int_{-\infty}^{+\infty} F'^2(y)\exp[g(y)]\,\mathrm{d}y/B. \tag{33}$$

Equations (7)–(9) give

$$\delta\bar{c}(y) = -y^3\delta\bar{\mu} - \frac{\bar{h}}{2a}\delta a \tag{34}$$

$$\delta\bar{c}'(y) = -3y^2\delta\bar{\mu} \tag{35}$$

$$\delta g(y) = g_{\bar{\mu}}(y)\delta\bar{\mu} + g_a(y)\delta a \tag{36}$$

where

$$g_{\bar{\mu}}(y) = -\frac{1}{4}y^4 + \bar{\tau}\bar{c}(y)y^3 \tag{37}$$

$$g_a(y) = \frac{\bar{h}}{2a}[\bar{\tau}\bar{c}(y) - y] - \frac{1}{2}\tau\bar{c}^2(y). \tag{38}$$

Substituting equations (35) and (36) into equation (29) yields

$$\delta L = \frac{G}{B} - L\cdot\frac{H}{B} \tag{39}$$

where

$$G = \int_{-\infty}^{+\infty} [W_{\bar{\mu}1}(y)\delta\bar{\mu} + W_{a1}(y)\delta a]\,\mathrm{d}y \tag{40}$$

$$H = \int_{-\infty}^{+\infty} [W_{\bar{\mu}2}(y)\delta\bar{\mu} + W_{a2}(y)\delta a]\,\mathrm{d}y \tag{41}$$

in which

$$W_{\bar{\mu}1}(y) = g_{\bar{\mu}}(y)F'^2(y)\exp[g(y)] \tag{42}$$

$$W_{a1}(y) = g_a(y)F'^2(y)\exp[g(y)] \tag{43}$$

$$W_{\bar{\mu}2}(y) = \{[1-\bar{\tau}\bar{c}'(y)]g_{\bar{\mu}}(y) + 3\bar{\tau}y^2\}F^2(y)\exp[g(y)] \tag{44}$$

$$W_{a2}(y) = \{[1-\bar{\tau}\bar{c}'(y)]g_a(y) - \tau\bar{c}'(y)\}F^2(y)\exp[g(y)]. \tag{45}$$

For the eigenvalue $\bar{\lambda}_1$, assume that the corresponding eigenvector is $\{v\} = [v_1, v_2, \ldots, v_n]^T$. Then with the help of equation (39), the variation of $\bar{\lambda}_1$ is

$$\delta\bar{\lambda}_1 = \bar{\lambda}_{\bar{\mu}1}\delta\bar{\mu} + \bar{\lambda}_{a1}\delta a - \bar{\lambda}_1(\bar{\lambda}_{\bar{\mu}2}\delta\bar{\mu} + \bar{\lambda}_{a2}\delta a) \tag{46}$$

where

$$\bar{\lambda}_1 = \frac{\{v\}^T[K]\{v\}}{\{v\}^T[M]\{v\}} \qquad \bar{\lambda}_{\bar{\mu}1} = \frac{\{v\}^T[K_{\bar{\mu}1}]\{v\}}{\{v\}^T[M]\{v\}} \qquad \bar{\lambda}_{a1} = \frac{\{v\}^T[K_{a1}]\{v\}}{\{v\}^T[M]\{v\}}$$
$$\bar{\lambda}_{\bar{\mu}2} = \frac{\{v\}^T[K_{\bar{\mu}2}]\{v\}}{\{v\}^T[M]\{v\}} \qquad \bar{\lambda}_{a2} = \frac{\{v\}^T[K_{a2}]\{v\}}{\{v\}^T[M]\{v\}}$$

in which $[K_{\bar{\mu}1}], [K_{a1}], [K_{\bar{\mu}2}]$ and $[K_{a2}]$ are the interpolating matrices of $\int_{-\infty}^{+\infty} W_{\bar{\mu}1}\,\mathrm{d}y$, $\int_{-\infty}^{+\infty} W_{a1}\,\mathrm{d}y$, $\int_{-\infty}^{+\infty} W_{\bar{\mu}2}\,\mathrm{d}y$ and $\int_{-\infty}^{+\infty} W_{a2}\,\mathrm{d}y$, respectively. These matrices are obtained by the Hermite interpolating method introduced in section 2. Putting equation (46) into equation (28) yields

$$\bar{\lambda}_1\delta a + a\bar{\lambda}_{\bar{\mu}1}\delta\bar{\mu} + a\bar{\lambda}_{a1}\delta a - a\bar{\lambda}_1(\bar{\lambda}_{\bar{\mu}2}\delta\bar{\mu} + \bar{\lambda}_{a2}\delta a) = 0. \tag{47}$$

In the following, we will optimize the system with equations (26) and (47).

3.2. The approach of tuning the system parameters

Generally, we will meet two problems when optimizing the output SNR with $\lambda_1 = \text{const}$ as described above. The first problem is that the evaluated system response will deviate from the prescribed one with increasing iterative number. As shown in figure 1, the curve near the points like A has large slope. Therefore, the second problem is that if Δa is chosen as the step size in the search, $\Delta\bar{\mu}$ cannot be precisely obtained near these points. In this case, there will be a large deviation of the evaluated system response speed from the prescribed one. For the first problem, we take a step by step compensation method to avoid the deviation. For the second one, the curve length increment Δs is taken as the step size in the search (see figure 1) instead of Δa, which can avoid the error near the points with large slopes. In the following, the two methods will be discussed in detail.

First of all, choose proper initial values a_0 and $\bar{\mu}_0$. Then evaluate the corresponding system response speed λ_{10} by equations (6) and (26). So the deviation from the prescribed λ_1 is

$$\varepsilon_0 = \lambda_1 - \lambda_{10} = \lambda_1 - a_0\bar{\lambda}_1(a_0, \bar{\mu}_0). \tag{48}$$

To compensate the deviation ε_0, equation (47) is rewritten as

$$(\bar{\lambda}_{10} + a_0\bar{\lambda}_{a10} - a_0\bar{\lambda}_{10}\bar{\lambda}_{a20})\Delta a + a_0(\bar{\lambda}_{\bar{\mu}10} - \bar{\lambda}_{10}\bar{\lambda}_{\bar{\mu}20})\Delta\bar{\mu} = \varepsilon_0. \tag{49}$$

From equation (47), the step size Δa_0 is

$$\begin{aligned}\Delta a_0 &= \Delta s\cos\beta \\ &= \frac{a_0(\bar{\lambda}_{10}\bar{\lambda}_{\bar{\mu}20} - \bar{\lambda}_{\bar{\mu}10})\Delta s}{\sqrt{(\bar{\lambda}_{10} + a_0\bar{\lambda}_{a10} - a_0\bar{\lambda}_{10}\bar{\lambda}_{a20})^2 + a_0^2(\bar{\lambda}_{10}\bar{\lambda}_{\bar{\mu}20} - \bar{\lambda}_{\bar{\mu}10})^2}}\end{aligned} \tag{50}$$

where β is shown in figure 1. $\Delta\bar{\mu}$ is given by equation (49), i.e.,

$$\Delta\bar{\mu}_0 = \frac{(\bar{\lambda}_{10} + a_0\bar{\lambda}_{a10} - a_0\bar{\lambda}_{10}\bar{\lambda}_{a20})\Delta a_0 - \varepsilon_0}{a_0(\bar{\lambda}_{10}\bar{\lambda}_{\bar{\mu}20} - \bar{\lambda}_{\bar{\mu}10})}. \tag{51}$$

Take

$$a_1 = a_0 + \Delta a_0 \qquad \bar{\mu}_1 = \bar{\mu}_0 + \Delta\bar{\mu}_0 \tag{52}$$

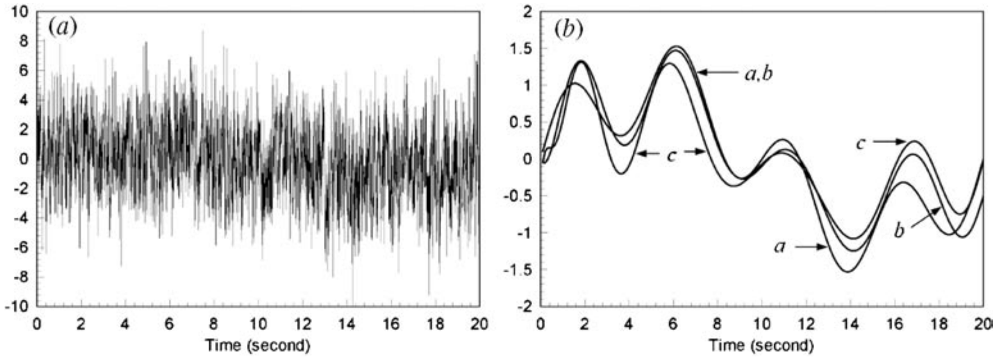

Figure 2. (*a*) The input signal spoiled by the coloured noise and (*b*) the input signal $h(t)$ and the recovery signals (*a*—the input signal; *b*—the recovery signal with the optimal parameters; *c*—the recovery signal with $a = 10$ and $\mu = 1000$).

as the parameters in the next step. When the computation is in the *i*th step, equations (48) and (50)–(52) could be rewritten by substituting *i* for the subscript 0 in these equations, i.e.,

$$\varepsilon_i = \lambda_1 - \lambda_{1i} = \lambda_1 - a_i\bar{\lambda}_1(a_i, \bar{\mu}_i) \tag{53}$$

$$\Delta a_i = \frac{a_i(\bar{\lambda}_{1i}\bar{\lambda}_{\bar{\mu}2i} - \bar{\lambda}_{\bar{\mu}1i})\Delta s}{\sqrt{(\bar{\lambda}_{1i} + a_i\bar{\lambda}_{a1i} - a_i\bar{\lambda}_{1i}\bar{\lambda}_{a2i})^2 + a_i^2(\bar{\lambda}_{1i}\bar{\lambda}_{\bar{\mu}2i} - \bar{\lambda}_{\bar{\mu}1i})^2}} \tag{54}$$

$$\Delta\bar{\mu}_i = \frac{(\bar{\lambda}_{1i} + a_i\bar{\lambda}_{a1i} - a_i\bar{\lambda}_{1i}\bar{\lambda}_{a2i})\Delta a_i - \varepsilon_i}{a_i(\bar{\lambda}_{1i}\bar{\lambda}_{\bar{\mu}2i} - \bar{\lambda}_{\bar{\mu}1i})} \tag{55}$$

$$a_{i+1} = a_i + \Delta a_i \qquad \bar{\mu}_{i+1} = \bar{\mu}_i + \Delta\bar{\mu}_i \tag{56}$$

which can be used to search for the maximal $\mathrm{SNR}_{\mathrm{out}}$ along the curve $\lambda_1 = \mathrm{const}$. Δs can be adjusted if it is needed. The initial values a_0 and $\bar{\mu}_0$ can also be changed in different regions (e.g. [0.01 0.1], [0.1 1], etc) in order to search for the maximal $\mathrm{SNR}_{\mathrm{out}}$ in the corresponding region if there are several limit values. After the optimal a and $\bar{\mu}$ are obtained, the optimal μ may be evaluated from equation (3).

4. The application of the method

In this section, we will simulate the nonlinear system with the optimized parameters by simulation software MATLAB®.

Consider an input signal $h(t) = \sin(0.1\pi t) + 0.6\sin(0.4\pi t)$ spoiled by Lorentzian coloured noise with the correlation time $\tau = 0.01$ and the intensity $D = 0.06$ (see figure 2(*a*)). Obviously, the signal period is 20 s.

The numerical simulation model is illustrated in figure 3. $h(t)$ is the input signal and $\Gamma(t)$ denotes the input white noise. The dashed-frame A simulates Lorentzian coloured noise (i.e., equation (2)), and B represents the nonlinear system (i.e., equation (1)), which was described in detail in [2].

In terms of the highest frequency of the input signal being 0.2 Hz, we take the system response speed λ_1 as 1.2 Hz. For the above analogue signal, the constant h in equation (1) should be determined before the system parameters are optimized. According to the method discussed in [23], h is approximately taken as 0.29. The optimal system parameter values for

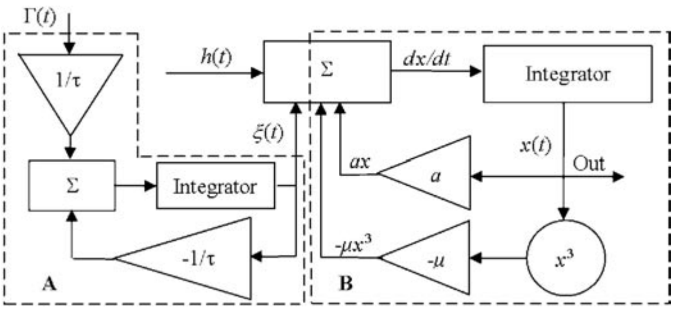

Figure 3. Functional block scheme for simulating the bistable system.

this magnitude and the mean SNR gain $M_{\mathrm{SNR\,gain}}$ [23] with the prescribed response speed could be evaluated by the method discussed in section 3. Their values are $a \approx 0.35$, $\mu \approx 10.21$ and $M_{\mathrm{SNR\,gain}} \approx 1.35$, respectively. The output sampling points can be recovered by the recovery formula, which is discussed in appendix C. Then join the recovery points by the method of Hermite curve fit [24] (see figure 2(*b*)). To show the effectiveness of optimization, choose $a = 10$ and $\mu = 1000$ at random and evaluate the corresponding $M_{\mathrm{SNR\,gain}} \approx 0.25$, which is much less than that with the above optimal parameter values. Figure 2(*b*) confirms the result. It shows that the recovery signal with the optimal parameter values is closer to the input signal.

5. Conclusions

In this paper, a simple but efficient method of evaluating the system response speed is introduced. Then we mainly discuss SR realized by adjusting the system parameters and put forward the method of tuning the parameters in detail. In fact, the maximal SNR gain can be obtained by solving an optimization problem with a prescribed system response speed. The numerical simulation confirms the effectiveness of the method. Finally, it is necessary to emphasize that the method is valid on the condition that the input signals are single-frequency signals or multi-frequency signals and the noise is white noise or Lorentzian coloured noise with short correlation time.

Acknowledgments

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Appendix A. The derivation of the system response speed

By analysis of an expansion in powers for the noise-correction time τ, a one-dimensional approximation of the Fokker–Planck equation is [21]

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}[c(x)\rho(x,t)] + D\frac{\partial^2}{\partial x^2}\left[\frac{\rho(x,t)}{1-\tau c'(x)}\right] \tag{A.1}$$

where $\rho(x,t)$ is the probability density, $c(x) = ax - \mu x^3 + h$ and $c'(x)$ is its derivative. The approximation is based on considering τ and D as small parameters (i.e., $\tau \ll 1$ and $D \ll 1$)

because the higher order terms of τ or D are neglected. Rescale the variables according to equation (3), and then equation (A.1) is recast as

$$\frac{\partial\rho(y,\bar{t})}{\partial\bar{t}} = -\frac{\partial}{\partial y}[\bar{c}(y)\rho(y,\bar{t})] + \frac{\partial^2}{\partial y^2}\left[\frac{\rho(y,\bar{t})}{1-\bar{\tau}\bar{c}'(y)}\right] \tag{A.2}$$

where $\bar{c}(y)$ and $\bar{c}'(y)$ are given in equations (7) and (8). The steady-state solution of equation (A.2) is given by [22]

$$\rho(y) = N|1-\bar{\tau}\bar{c}'(y)|\exp[g(y)] \tag{A.3}$$

where $g(y)$ is given in equation (9) and the constant N is determined by the normalization condition of the probability density $\rho(x)$,

$$N = \left[\sqrt{\frac{D}{a}}\int_{-\infty}^{+\infty}|1-\bar{\tau}\bar{c}'(y)|\exp[g(y)]\,\mathrm{d}y\right]^{-1}. \tag{A.4}$$

Under the condition $1-\bar{\tau}\bar{c}'(y) > 0$, an approximate solution of equation (A.2) may be assumed as

$$\rho(y,\bar{t}) = \Psi(y,\bar{t})[1-\bar{\tau}\bar{c}'(y)]\exp\left[\tfrac{1}{2}g(y)\right]. \tag{A.5}$$

Putting equation (A.5) into equation (A.2) yields

$$\frac{\partial\Psi(y,\bar{t})}{\partial\bar{t}}[1-\bar{\tau}\bar{c}'(y)] = \frac{\partial^2\Psi(y,\bar{t})}{\partial y^2} - V(y)\Psi(y,\bar{t}) \tag{A.6}$$

where

$$V(y) = \tfrac{1}{2}g''(y) + \tfrac{1}{4}g'^2(y). \tag{A.7}$$

Obviously, the right-hand side of equation (A.6) can be considered as a self-conjugate differential operator acting on $\Psi(y,\bar{t})$. Define $\Psi(y,\bar{t})$ as

$$\Psi(y,\bar{t}) = u(y)\exp[-\bar{\lambda}\bar{t}]. \tag{A.8}$$

Substituting $\Psi(y,\bar{t})$ into equation (A.6) yields

$$-\bar{\lambda}u(y)[1-\bar{\tau}\bar{c}'(y)] = u''(y) - V(y)u(y). \tag{A.9}$$

Multiplying both sides of equation (A.9) by $u(y)$ and integrating it from minus infinity to infinity with respect to boundary conditions $\lim_{y\to\pm\infty}u(y) = \lim_{y\to\pm\infty}u'(y) = 0$, equation (A.9) can be transformed into a Rayleigh quotient form, i.e.,

$$\bar{\lambda} = \underset{u\neq 0}{\mathrm{st}}\,\frac{\int_{-\infty}^{+\infty}[V(y)u^2(y)+u'^2(y)]\,\mathrm{d}y}{\int_{-\infty}^{+\infty}[1-\bar{\tau}\bar{c}'(y)]u^2(y)\,\mathrm{d}y}. \tag{A.10}$$

Assume

$$u(y) = F(y)\exp\left[\tfrac{1}{2}g(y)\right] \tag{A.11}$$

so the integrated function is

$$\begin{aligned} V(y)u^2(y)+u'^2(y) &= \left\{\left[\tfrac{1}{2}g''(y)+\tfrac{1}{4}g'^2(y)\right]F^2(y) + \left[F'(y)+\tfrac{1}{2}F(y)g'(y)\right]^2\right\}\exp[g(y)] \\ &= \left\{\tfrac{1}{2}\left[g'(y)F^2(y)\right]' + \tfrac{1}{2}g'^2(y)F^2(y) + F'^2(y)\right\}\exp[g(y)]. \end{aligned} \tag{A.12}$$

Since

$$\begin{aligned} &\int_{-\infty}^{+\infty}[g'(y)F^2(y)]'\exp[g(y)]\,\mathrm{d}y \\ &\quad = g'(y)F^2(y)\exp[g(y)]|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}g'^2(y)F^2(y)\exp[g(y)]\,\mathrm{d}y \\ &\quad = -\int_{-\infty}^{+\infty}g'^2(y)F^2(y)\exp[g(y)]\,\mathrm{d}y \end{aligned} \tag{A.13}$$

then

$$\int_{-\infty}^{+\infty} [V(y)u^2(y) + u'^2(y)]\,\mathrm{d}y = \int_{-\infty}^{+\infty} F'^2(y)\exp[g(y)]\,\mathrm{d}y. \tag{A.14}$$

Therefore, equation (A.10) is changed as equation (6).

Appendix B. The definition of the SNR gain

In this appendix, define the integrals

$$f_i(\bar{\mu}, \bar{h}, \bar{\tau}) = \int_{-\infty}^{+\infty} y^i |1 - \bar{\tau}\bar{c}'(y)| \exp[g(y)]\,\mathrm{d}y \qquad i = 1, 2. \tag{B.1}$$

Then equation (A.4) is changed as

$$N = \left[\sqrt{\frac{D}{a}} f_0(\bar{\mu}, \bar{h}, \bar{\tau})\right]^{-1}. \tag{B.2}$$

The asymptotic output SNR is [23]

$$\mathrm{SNR}_{\mathrm{out}} = \lim_{t\to\infty} \frac{E^2[x(t)]}{D[x(t)]} \tag{B.3}$$

where $E[\cdot]$ and $D[\cdot]$ are the statistical average and variance of the stochastic variable, respectively. From equations (3), (A.3) and (B.2), we have

$$\begin{aligned} E[x] &= \int_{-\infty}^{+\infty} x\rho(x)\,\mathrm{d}x = N\int_{-\infty}^{+\infty} x|1 - \bar{\tau}\bar{c}'(y)| \exp[g(y)]\,\mathrm{d}x \\ &= N\frac{D}{a}\int_{-\infty}^{+\infty} y|1 - \bar{\tau}\bar{c}'(y)| \exp[g(y)]\,\mathrm{d}y = \sqrt{\frac{D}{a}}\frac{f_1(\bar{\mu}, \bar{h}, \bar{\tau})}{f_0(\bar{\mu}, \bar{h}, \bar{\tau})} \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} E[x^2] &= N\int_{-\infty}^{+\infty} x^2|1 - \bar{\tau}\bar{c}'(y)| \exp[g(y)]\,\mathrm{d}x \\ &= N\sqrt{\frac{D^3}{a^3}}\int_{-\infty}^{+\infty} y^2|1 - \bar{\tau}\bar{c}'(y)| \exp[g(y)]\,\mathrm{d}y = \frac{Df_2(\bar{\mu}, \bar{h}, \bar{\tau})}{af_0(\bar{\mu}, \bar{h}, \bar{\tau}}. \end{aligned} \tag{B.5}$$

With the help of equations (B.4) and (B.5), the variance is

$$D[x] = E[x^2] - (E[x])^2 = \frac{Df_2(\bar{\mu}, \bar{h}, \bar{\tau})}{af_0(\bar{\mu}, \bar{h}, \bar{\tau})} - \frac{Df_1^2(\bar{\mu}, \bar{h}, \bar{\tau})}{af_0^2(\bar{\mu}, \bar{h}, \bar{\tau})}. \tag{B.6}$$

Then from equations (B.4) and (B.6), the output SNR is

$$\mathrm{SNR}_{\mathrm{out}} = \frac{f_1^2(\bar{\mu}, \bar{h}, \bar{\tau})}{f_0(\bar{\mu}, \bar{h}, \bar{\tau})f_2(\bar{\mu}, \bar{h}, \bar{\tau}) - f_1^2(\bar{\mu}, \bar{h}, \bar{\tau})} = \mathrm{SNR}_{\mathrm{out}}(\bar{\mu}, \bar{h}, \bar{\tau}). \tag{B.7}$$

Define the input SNR as

$$\mathrm{SNR}_{\mathrm{in}} = h^2/D = a\bar{h}^2 \tag{B.8}$$

so the SNR gain is given by

$$\mathrm{SNR}_{\mathrm{gain}} = \mathrm{SNR}_{\mathrm{out}}(\bar{\mu}, \bar{h}, \bar{\tau})D/h^2. \tag{B.9}$$

Appendix C. The derivation of the recovery formula [24]

Consider the system

$$\dot{x}(t) = f(x) + h(t) + \xi(t) \tag{C.1}$$

where

$$f(x) = ax(t) - \mu x^3(t). \tag{C.2}$$

If $|\dot{x}| \ll |h|$, there will be

$$h(t) \approx -f(x) - \xi(t). \tag{C.3}$$

Take the mean value for this equation and note that $E[\xi(t)] = 0$, then

$$\tilde{h} = E[-f(x)] \approx -f(\tilde{x}). \tag{C.4}$$

Here, $\tilde{h} = E$ and $\tilde{x} = E[x]$. With this result, the condition $|\dot{x}| \ll |h|$ becomes

$$|h'(t)| \ll |f'(\tilde{x})h(t)|. \tag{C.5}$$

Obviously, when $h(t)$ is close to zero, the condition (C.5) is not satisfied and the output cannot be recovered by equation (C.4).

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